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ON ELASTIC, WORKHARDENING SOLIDS*

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Abstract—A set of integral equations is established for the rate of change of the stress field that corresponds to a given rate of loading from a given elasto–plastic state in an elastic, workhardening solid. An approximate method is developed for the solution of these integral equations in three- and two-dimensional elasto–plastic problems. The results are applied to one-dimensional structures, such as continuous beams, and explicit solutions are obtained.

1. INTRODUCTION

THE stress and the strain fields at any stage in a quasistatic process of the loading of an elastic, workhardening solid may be determined by the repeated solution of the following problem: To find the rates of change of the stress and the strain fields that correspond to a given rate of loading from a given elasto–plastic state of the solid. In the present paper, a novel approach is suggested for the formulation and solution of this basic problem. The stress-rates are regarded as the response of the elastic solid to the rate of loading and the (as yet unknown) plastic strain-rates. Using the concept of the influence function (Green's function) the stress-rates are then expressed in terms of the rate of loading by means of a set of integral equations.

Such an integral equation formulation, while of obvious theoretical interest, does not lend itself to an easy numerical calculation, since the exact Green's function is, in general, very difficult to obtain. To remedy this, approximate methods for the solution of three- and two-dimensional elasto–plastic problems are developed. Based on the Ritz method, a system of integral equations with degenerate kernels is established. These equations can approximate the basic problem to any desired degree of accuracy. Finally, the method is applied to one-dimensional elasto–plastic structures, such as continuous beams, yielding an exact solution for the rate of bending moments.

2. STATEMENT OF PROBLEM AND BASIC EQUATIONS

Consider a solid that consists of an elastic, workhardening material and occupies a volume V with a regular surface S. Assume that the surface tractions T_i are prescribed on the part S_T of the surface S, and the surface displacements $u_i \equiv 0$ are defined on the remainder S_u of S in such a manner that the response of the solid to arbitrarily given body forces F_i or surface tractions T_i entails no rigid-body motion.‡ Let the solid be in the stress-

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‡ All quantities are referred to a fixed rectangular Cartesian coordinate system x_i; $i = 1, 2, 3$.

free, virgin state for $T_i \equiv 0$, $F_i \equiv 0$. Denote by u_i the infinitesimal displacements of the material points of the solid from the virgin state to the state of stress τ_{ij} which is caused by the surface tractions T_i and the body forces F_i. Assume that the displacement-gradient $\partial u_i/\partial x_j$ is so small that it is unnecessary to distinguish between Eulerian and Lagrangian variables.

Under these conditions and when the response of the solid to surface tractions T_i and body forces F_i is purely elastic, i.e. entails no plastic deformations, the displacement field of the solid can be formulated using Green's function (e.g., Pearson [1], Chapter VI). Let $G_i^{(\beta)}(\mathbf{x};\boldsymbol{\xi})$ denote the displacement in the i-direction at point $\mathbf{x}(x_1, x_2, x_3)$, due to the action of a unit load in the β-direction at point $\boldsymbol{\xi}(\xi_1, \xi_2, \xi_3)$, when the tractions T_i on S_T, and the displacements $G_i^{(\beta)}(\mathbf{x};\boldsymbol{\xi})$ for $\mathbf{x}$ on S_u are identically zero. Green's function $G_i^{(\beta)}$ is symmetric and satisfies the following field equations:

$$C_{ijkl}G_{k,lj}^{(\beta)} + \delta_{i\beta}\delta(\mathbf{x}-\boldsymbol{\xi}) = 0 \quad \text{in } V, \tag{2.1}$$

$$C_{ijkl}G_{k,l}^{(\beta)}n_j = 0 \quad \text{on } S_T, \quad \text{and} \quad G_k^{(\beta)} = 0 \quad \text{on } S_u, \tag{2.2}$$

where $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + 2\mu\delta_{ik}\delta_{jl}$, Roman subscripts following a comma indicate partial differentiation with respect to the corresponding components of the vector $\mathbf{x}$, λ and μ are the Lamé constants, δ_{ij} is the Kronecker delta, and $\delta(\mathbf{x}-\boldsymbol{\xi})$ denotes the Dirac delta function. In (2.2), n_i is the exterior unit normal to S_T.

The purely elastic response of the solid to a given load T_i and F_i may now be formulated as follows:

$$u_i(\mathbf{x}) = \int_{S_T} G_i^{(\beta)}(\mathbf{x};\boldsymbol{\xi})T_\beta(\boldsymbol{\xi})\,\mathrm{d}S + \int_V G_i^{(\beta)}(\mathbf{x};\boldsymbol{\xi})F_\beta(\boldsymbol{\xi})\,\mathrm{d}V, \tag{2.3}$$

$$\tau_{ij} = C_{ijkl}\varepsilon_{kl} = C_{ijkl}u_{k,l}, \tag{2.4}$$

where the integrations are to be carried out with respect to $\boldsymbol{\xi}$, $\mathbf{x}$ being held constant, and $\varepsilon_{kl} = \frac{1}{2}(u_{k,l}+u_{l,k})$ is the strain tensor.

Now assume that the solid is at an elasto–plastic state under the action of a known set of surface tractions T_i on S_T and body forces F_i in V. Consider the rate of loading defined by the rates of the surface traction $\dot{T}_i$ on S_T and the body force $\dot{F}_i$ in V. Let the corresponding rate of the plastic strain be denoted by $\dot{\varepsilon}''_{ij}$ and the rate of elastic strain by $\dot{\varepsilon}'_{ij}$. We have

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}'_{ij} + \dot{\varepsilon}''_{ij},$$

$$\dot{\tau}_{ij} = C_{ijkl}\dot{\varepsilon}'_{kl} = C_{ijkl}(\dot{u}_{k,l} - \dot{\varepsilon}''_{kl}). \tag{2.5}$$

The stress-rate $\dot{\tau}_{ij}$ must satisfy the equations of equilibrium and the boundary conditions

$$\dot{\tau}_{ij,j} + \dot{F}_i = 0 \quad \text{in } V,$$
$$\dot{\tau}_{ij}n_j = \dot{T}_i \quad \text{on } S_T, \tag{2.6a}$$

or

$$C_{ijkl}\dot{u}_{k,lj} + (\dot{F}_i - C_{ijkl}\dot{\varepsilon}''_{kl,j}) = 0 \quad \text{in } V,$$
$$C_{ijkl}\dot{u}_{k,l}n_j = (\dot{T}_i + C_{ijkl}\dot{\varepsilon}''_{kl}n_j) \quad \text{on } S_T. \tag{2.6b}$$

Equations (2.6b) imply that, when the rate of the plastic strain is supposed to be known, the solution of the incremental elasto–plastic problem can be obtained by solving an elastic problem where, instead of the true rates of loading $\dot{T}_i$ and $\dot{F}_i$, the following fictitious rates of surface traction $T_i^{*\cdot}$ and body force $F_i^{*\cdot}$ are considered:

$$\begin{aligned} T_i^{*\cdot} &= \dot{T}_i + C_{ijkl}\dot{\varepsilon}''_{kl}n_j \quad \text{on } S_T, \\ F_i^{*\cdot} &= \dot{F}_i - C_{ijkl}\dot{\varepsilon}''_{kl,j} \quad \text{in } V. \end{aligned} \tag{2.7a}$$

The rate of the displacement vector $\dot{u}_i$ may, therefore, be written as

$$\begin{aligned} \dot{u}_i(\mathbf{x}) = &\int_{S_T} G_i^{(\beta)}(\mathbf{x};\boldsymbol{\xi})\dot{T}_\beta(\boldsymbol{\xi})\,\mathrm{d}S + \int_V G_i^{(\beta)}(\mathbf{x};\boldsymbol{\xi})\dot{F}_\beta(\boldsymbol{\xi})\,\mathrm{d}V \\ &+ \int_{S_T} C_{\beta jkl}G_i^{(\beta)}(\mathbf{x};\boldsymbol{\xi})\dot{\varepsilon}''_{kl}(\boldsymbol{\xi})n_j\,\mathrm{d}S \\ &- \int_V C_{\beta\alpha kl}G_i^{(\beta)}(\mathbf{x};\boldsymbol{\xi})\dot{\varepsilon}''_{kl,\alpha}(\boldsymbol{\xi})\,\mathrm{d}V. \end{aligned} \tag{2.7b}$$

The first two terms in the right side of this equation would represent the rate of displacement due to the additional loading if the solid were to respond to such loading in a purely elastic way. We designate this part of $\dot{u}_i$ by $u_i^{0\cdot}$ and with the aid of the divergence theorem reduce the above equation to the following form:

$$\dot{u}_i(\mathbf{x}) = u_i^{0\cdot}(\mathbf{x}) + \int_V C_{\alpha\beta kl}G_{i,\beta}^{(\alpha)}(\mathbf{x};\boldsymbol{\xi})\dot{\varepsilon}''_{kl}(\boldsymbol{\xi})\,\mathrm{d}V, \tag{2.8a}$$

where Greek subscripts following a comma indicate partial differentiation with respect to the corresponding components of the vector $\boldsymbol{\xi}$.

Since $G_i^{(\alpha)}(\mathbf{x};\boldsymbol{\xi})$ represents the displacement at point $\mathbf{x}$ due to a unit load at point $\boldsymbol{\xi}$ in the α-direction, the term $G_{i,\beta}^{(\alpha)}(\mathbf{x};\boldsymbol{\xi})$ corresponds to the displacement vector at $\mathbf{x}$ due to various nuclei of strain [2] at point $\boldsymbol{\xi}$. For example, with $\alpha = \beta = 1$, $G_{i,\beta}^{(\alpha)}(\mathbf{x};\boldsymbol{\xi})$ defines the displacement field of a "double force without moment" applied along the 1-axis at $\boldsymbol{\xi}$, while with $\alpha = 1$, $\beta = 2$ it denotes the displacement field of a "double force with moment" applied along the 1-direction at point $\boldsymbol{\xi}$. Writing the integrand in equation (2.8a) in an expanded form, we have

$$C_{\alpha\beta kl}G_{i,\beta}^{(\alpha)}(\mathbf{x};\boldsymbol{\xi})\dot{\varepsilon}''_{kl}(\boldsymbol{\xi}) = \lambda\left[\frac{\partial G_i^{(1)}}{\partial\xi_1} + \frac{\partial G_i^{(2)}}{\partial\xi_2} + \frac{\partial G_i^{(3)}}{\partial\xi_3}\right]\dot{\varepsilon}''_{kk} + 2\mu G_{i,\beta}^{(\alpha)}\dot{\varepsilon}''_{\alpha\beta}.$$

The expression inside of the brackets in the right side of this equation corresponds to the displacement field of a "center of dilatation" applied at point $\boldsymbol{\xi}$. We shall assume that $\dot{\varepsilon}''_{ii} = 0$ and, therefore, reduce equation (2.8a) to

$$\dot{u}_i(\mathbf{x}) = u_i^{o\cdot}(\mathbf{x}) + 2\mu\int_V G_{i,\beta}^{(\alpha)}(\mathbf{x};\boldsymbol{\xi})\dot{\varepsilon}''_{\alpha\beta}(\boldsymbol{\xi})\,\mathrm{d}V. \tag{2.8b}$$

We now differentiate both sides of equation (2.8b) with respect to x_j and obtain

$$\dot{u}_{i,j}(\mathbf{x}) = u_{i,j}^{0\cdot}(\mathbf{x}) + 2\mu\int_V G_{i,\beta j}^{(\alpha)}(\mathbf{x};\boldsymbol{\xi})\dot{\varepsilon}''_{\alpha\beta}(\boldsymbol{\xi})\,\mathrm{d}V \tag{2.8c}$$

which represents the rate of the displacement-gradient at point $\mathbf{x}$. The stress-rate $\tau_{ij}^{\cdot}$ may now be obtained by substituting from (2.8c) into equation (2.5)

$$\tau_{ij}^{\cdot}(\mathbf{x}) = \tau_{ij}^{0\cdot}(\mathbf{x})+2\mu\int_V C_{ijkl}G_{k,\beta l}^{(\alpha)}(\mathbf{x};\boldsymbol{\xi})\varepsilon_{\alpha\beta}^{\prime\prime\cdot}(\boldsymbol{\xi})\,\mathrm{d}V-2\mu\varepsilon_{ij}^{\prime\prime\cdot}(\mathbf{x}), \tag{2.9}$$

where $\tau_{ij}^{0\cdot} = C_{ijkl}u_{k,l}^{0\cdot}$, and $\varepsilon_{ii}^{\prime\prime\cdot} = 0$.

The first-term in the right side of equation (2.9) would be the stress-rate if the solid were to deform elastically only. The term $G_{k,\beta l}^{(\alpha)}(\mathbf{x};\boldsymbol{\xi})$, on the other hand, represents the gradient of the kth component of the displacement vector in the l-direction at $\mathbf{x}$ due to a nucleus of strain associated with the α, β-directions at $\boldsymbol{\xi}$. Thus the term $2\mu C_{ijkl}G_{k,\beta l}^{(\alpha)}(\mathbf{x};\boldsymbol{\xi})$ represents the stress tensor at $\mathbf{x}$ due to various nuclei of strain applied at point $\boldsymbol{\xi}$. We use the notation $L_{ij}^{(\alpha\beta)}(\mathbf{x};\boldsymbol{\xi})$ to designate this influence function (Green's function) and note the following reciprocity relation:

$$L_{ij}^{(\alpha\beta)}(\mathbf{x};\boldsymbol{\xi}) = L_{\alpha\beta}^{(ij)}(\boldsymbol{\xi};\mathbf{x}).$$

Equation (2.9) now reduces to

$$\tau_{ij}^{\cdot}+2\mu\varepsilon_{ij}^{\prime\prime\cdot}-\int_V L_{ij}^{(\alpha\beta)}(\mathbf{x};\boldsymbol{\xi})\varepsilon_{\alpha\beta}^{\prime\prime\cdot}(\boldsymbol{\xi})\,\mathrm{d}V = \tau_{ij}^{0\cdot}(\mathbf{x}) \tag{2.10}$$

which defines the rate of plastic strain $\varepsilon_{ij}^{\prime\prime\cdot}$ in terms of the elastic properties of the solid and the rates of the stress tensors $\tau_{ij}^{\cdot}$ and $\tau_{ij}^{0\cdot}$. To complete the formulation of the problem a phenomenological equation which relates the stress-rate $\tau_{ij}^{\cdot}$ to the plastic strain-rate $\varepsilon_{ij}^{\prime\prime\cdot}$ must now be postulated.

Somewhat similar lines of reasoning to that presented above has been used by Nabarro [3] to obtain the stress field of certain prescribed dislocations in an infinitely extended elastic body, and by Mura [4] in treating the problem of the continuous distribution of moving dislocations. These ideas have also been employed by Lin† and Ito [5] in calculating the latent elastic strain energy due to given plastic strains in polycrystals. In these studies, however, the plastic strains are *a priori* given and the stresses are then defined in terms of these strains. In the present study, on the other hand, both the stress-rate $\tau_{ij}^{\cdot}$ and the plastic strain-rate $\varepsilon_{ij}^{\prime\prime\cdot}$ are unknowns which are to be calculated using (2.10) and the appropriate constitutive equations.

3. ELASTIC, WORKHARDENING SOLIDS

When the solid consists of an elastic, workhardening material, we may write (e.g., Naghdi [6])

$$\varepsilon_{ij}^{\prime\prime\cdot} = \begin{cases} A_{ijkl}\tau_{kl}^{\cdot} & \text{for loading} \\ 0 & \text{for unloading,} \end{cases} \tag{3.1}$$

where the fourth order tensor A_{ijkl} may depend on the considered plastic state;

$$A_{ijkl} = A_{ijkl}(\tau_{mn}, \varepsilon_{mn}^{\prime\prime}). \tag{3.2}$$

† Further references to the work of Lin and his coworkers are cited in [5].

The plastic state is generally characterized by the vanishing of a yield function

$$f = f(\tau_{mn}, \varepsilon''_{mn}, \varkappa)$$

of the components of stress, plastic strain, and a workhardening parameter $\varkappa$. The sign of f is chosen in such a manner that $f < 0$ defines elastic states. The law (3.1) may then be written as follows:

$$\begin{aligned} \dot{\varepsilon}''_{ij} &= \frac{1}{D}\frac{\partial f}{\partial \tau_{ij}}\frac{\partial f}{\partial \tau_{kl}}\dot{\tau}_{kl} \quad \text{if } f = 0 \quad \text{and} \quad \frac{\partial f}{\partial \tau_{ij}}\dot{\tau}_{ij} \geq 0 \\ \dot{\varepsilon}''_{ij} &= 0 \quad \text{if} \begin{cases} f < 0 & \text{or} \\ f = 0 & \text{and} \quad \dfrac{\partial f}{\partial \tau_{ij}}\dot{\tau}_{ij} < 0, \end{cases} \end{aligned} \tag{3.3}$$

where, for workhardening solids,

$$D = -\frac{\partial f}{\partial \tau_{ij}}\frac{\partial f}{\partial \varepsilon''_{ij}} > 0.$$

From (3.1) and (3.3), we thus have

$$A_{ijkl} = \begin{cases} \dfrac{1}{D}\dfrac{\partial f}{\partial \tau_{ij}}\dfrac{\partial f}{\partial \tau_{kl}} & \text{in } V'' \\ 0 & \text{in } V', \end{cases} \tag{3.4}$$

where V'' defines the collection of the elements which are instantaneously yielding, and V' defines the collection of the remaining elements of solid that are instantaneously unloading or continuing to deform elastically.

With the aid of the stress–strain law (3.1), equation (2.5) reduces to

$$\dot{\tau}_{ij} = \begin{cases} C_{ijkl}(\dot{u}_{k,l} - A_{klmn}\dot{\tau}_{mn}) & \text{in } V'' \\ C_{ijkl}\dot{u}_{k,l} & \text{in } V'. \end{cases}$$

Noting that $\dot{\varepsilon}''_{ii} = 0$, this equation may be written as

$$C_{ijkl}\dot{u}_{k,l} = \begin{cases} \dot{\tau}_{kl}(\delta_{ik}\delta_{jl} + 2\mu A_{ijkl}) & \text{in } V'' \\ \dot{\tau}_{ij} & \text{in } V'. \end{cases} \tag{3.5}$$

With the aid of the relation (3.5), equation (2.10) becomes

$$(\delta_{ik}\delta_{jl} + 2\mu A_{ijkl}(\mathbf{x}))\dot{\tau}_{kl}(\mathbf{x}) = \dot{\tau}^0_{ij}(\mathbf{x}) + \int_{V''} L^{(\alpha\beta)}_{ij}(\mathbf{x};\boldsymbol{\xi})A_{\alpha\beta kl}(\boldsymbol{\xi})\dot{\tau}_{kl}(\boldsymbol{\xi})\,\mathrm{d}V. \tag{3.6}$$

For a given plastic state, the stress τ_{ij} and plastic strain ε''_{ij} are known functions of position $\mathbf{x}$. From (3.2), A_{ijkl} is thus defined as a function of $\mathbf{x}$ in V''; it is zero in V'. When, for a given rate of loading, no unloading takes place, then the nonsingular, 9 by 9 symmetric matrix $(\delta_{ik}\delta_{jl} + 2\mu A_{ijkl})$ may be inverted to yield

$$K_{ijkl}(\mathbf{x}) = \begin{cases} (\delta_{ik}\delta_{jl} + 2\mu A_{ijkl}(\mathbf{x}))^{-1} & \text{in } V'' \\ \delta_{ik}\delta_{jl} & \text{in } V', \end{cases} \tag{3.7}$$

where the superposed-1 defines the inverse. In this case, (3.6) may be written as

$$\dot{\tau}_{mn}(\mathbf{x}) = K_{mnij}(\mathbf{x})\left\{\dot{\tau}^0_{ij}(\mathbf{x}) + \int_{V''} L^{(\alpha\beta)}_{ij}(\mathbf{x};\boldsymbol{\xi})A_{\alpha\beta kl}(\boldsymbol{\xi})\dot{\tau}_{kl}(\boldsymbol{\xi})\,\mathrm{d}V\right\}. \tag{3.8}$$

For a given plastic state and plastic strain-rate $\varepsilon''^{\cdot}_{ij}$, the rates of displacement vector $u^{\cdot}_i$ and stress tensor $\tau^{\cdot}_{ij}$ are uniquely defined by, and may be calculated from equations (2.8b) and (2.9), respectively. Therefore, it appears reasonable to consider the stress-rate $\tau^{\cdot}_{ij}$ as the unknown; it is defined by the integral equation (3.6). In this equation the kernel $L^{(\alpha\beta)}_{ij}(\mathbf{x};\boldsymbol{\xi}) \equiv 2\mu C_{ijkl}G^{(\alpha)}_{k,\beta l}(\mathbf{x};\boldsymbol{\xi})$, which represents the stress tensor at $\mathbf{x}$ due to various nuclei of strain at $\boldsymbol{\xi}$, defines the elastic properties of the solid and may be calculated (at least approximately) for a given solid with a prescribed geometry. Moreover, $\tau^{0\cdot}_{ij}$ may be calculated for a known rate of loading using the elasticity theory. The quantity A_{ijkl} depends on the state of stress, the plastic strain tensor, the position of the considered element, and also on whether an element undergoes loading or unloading. Formally, equation (3.6) states that, for a given rate of loading, the solution of the elasto–plastic problem may be obtained by solving a nonhomogeneous anisotropic, elastic problem; this problem is linear only if no unloading takes place, in which case equation (3.8) may be used.

The calculation of an exact Green's function $L^{(\alpha\beta)}_{ij}(\mathbf{x};\boldsymbol{\xi})$ for two- and three-dimensional problems is, in general, very difficult if not impossible. In addition, even if such an exact Green's function is available, the system of integral equations (3.6) can not be readily solved unless this Green's function is degenerate, in which case an exact solution may be obtained. This is the case for some one-dimensional structures, for example continuous beams and rigid frames (see Section 4). For most other cases, one is forced into using some approximate procedures for the solution of the system (3.6).

One approach would be to replace the kernels in (3.6) with some approximate, degenerate kernels;

$$L^{(\alpha\beta)}_{ij}(\mathbf{x};\boldsymbol{\xi}) = \sum_{n=1}^{N} \psi^{(n)}_{ij}(\mathbf{x})\psi^{(n)}_{\alpha\beta}(\boldsymbol{\xi}). \tag{3.9}$$

This is always possible if we have available a sequence of functions

$$\varphi^{(1)}_i(\mathbf{x}),\quad \varphi^{(2)}_i(\mathbf{x}),\ldots,\varphi^{(N)}_i(\mathbf{x});\qquad i = 1,2,3, \tag{3.10a}$$

which are energy orthonormal (i.e. for each m and n we have

$$\int_V C_{ijkl}\varphi^{(n)}_{i,j}(\mathbf{x})\varphi^{(m)}_{k,l}(\mathbf{x})\,\mathrm{d}V = \delta_{mn},$$

where δ_{mn} is the Kronecker delta) and which satisfy all the geometrical and continuity requirements on the displacement field of the considered solid. In addition, for any N, the functions (3.10a) must be linearly independent and form a sequence which is complete in energy (see Mikhlin [7]) with respect to the class of functions v_i that satisfy the geometrical boundary conditions of the problem and possess the required degree of differentiability.† Now, using equation (2.1), (or equivalently the Ritz method [7]), we obtain

$$G^{(\alpha}_{k,\beta l}(\mathbf{x};\boldsymbol{\xi}) \approx -\sum_{n=1}^{N} \varphi^{(n)}_{k,l}(\mathbf{x})\varphi^{(n)}_{\alpha,\beta}(\boldsymbol{\xi}).$$

† The completeness of the sequence $\varphi^{(n)}_i$ with respect to energy implies that a function v_i can be approximated by

$$v^N_i = \sum_{m=1}^{N} a_m\varphi^{(m)}_i(\mathbf{x}),$$

such that the energy difference

$$\int_V C_{ijkl}[v_{i,j}v_{k,l} - v^N_{i,j}v^N_{k,l}]\,\mathrm{d}V$$

becomes arbitrarily small for sufficiently large N.

The functions in the right-hand side of (3.9) are thus defined by

$$\psi_{ij}^{(n)}(\mathbf{x})\psi_{\alpha\beta}^{(n)}(\xi) = -2\mu C_{ijkl}\varphi_{k,l}^{(n)}(\mathbf{x})\varphi_{\alpha,\beta}^{(n)}(\xi); \qquad \text{(no sum on } n). \tag{3.10b}$$

Substitution from (3.9) into (3.6) now results in the following system of integral equations:

$$(\delta_{ik}\delta_{jl}+2\mu A_{ijkl}(\mathbf{x}))\dot{\tau}_{kl}(\mathbf{x}) = \dot{\tau}^0_{ij}(\mathbf{x})+\sum_{n=1}^{N}\psi_{ij}^{(n)}(\mathbf{x})\int_{V''}\psi_{\alpha\beta}^{(n)}(\xi)A_{\alpha\beta\gamma\zeta}(\xi)\dot{\tau}_{\gamma\zeta}(\xi)\,\mathrm{d}V;$$

$$\mathrm{i,j,k,l},\alpha,\beta,\gamma,\zeta = 1,2,3, \tag{3.11}$$

which can be solved directly if no unloading occurs.

To this end, let

$$B^{(n)} = \int_{V''}\psi_{\alpha\beta}^{(n)}(\xi)A_{\alpha\beta\gamma\zeta}(\xi)\dot{\tau}_{\gamma\zeta}(\xi)\,\mathrm{d}V$$

and from (3.7) and (3.11) obtain

$$\dot{\tau}_{ij}(\mathbf{x}) = K_{ijkl}(\mathbf{x})\left[\dot{\tau}^0_{kl}(\mathbf{x})+\sum_{n=1}^{N}B^{(n)}\psi_{kl}^{(n)}(\mathbf{x})\right]. \tag{3.12}$$

Multiplying both sides of (3.12) by $\psi_{pq}^{(m)}(\mathbf{x})A_{pqij}(\mathbf{x})$, and integrating the results over V'' with respect to $\mathbf{x}$ yield

$$\sum_{n=1}^{N}(\delta_{mn}-\Psi_{mn})B^{(n)} = f_m; \qquad m = 1,2,\ldots,N, \tag{3.13}$$

where

$$\Psi_{mn} = \int_{V''}\psi_{pq}^{(m)}(\mathbf{x})A_{pqij}(\mathbf{x})K_{ijkl}(\mathbf{x})\psi_{kl}^{(n)}(\mathbf{x})\,\mathrm{d}V,$$

and

$$f_m = \int_{V''}\psi_{pq}^{(m)}(\mathbf{x})A_{pqij}(\mathbf{x})K_{ijkl}(\mathbf{x})\dot{\tau}^0_{kl}(\mathbf{x})\,\mathrm{d}V.$$

The system of linear equations (3.13) may be solved for $B^{(n)}$ which may then be substituted into (3.11), yielding an approximate expression for $\dot{\tau}_{kl}$.

Note that A_{ijkl} is zero if unloading takes place, or if an element continues to deform only elastically. The possibility of unloading complicates the analysis to some extent, since Ψ_{mn} and f_m in (3.13) depend on the plastic region V'' which, if unloading occurs, is not known *a priori*. In this case one might use an iterative method [8] or other numerical techniques. In any event the question of whether, for a given rate of loading, an element that has been in a plastic state unloads or not can only be resolved *a posteriori*; i.e. after the stress-rate $\dot{\tau}_{ij}$ is known.

In the next section, we shall specialize the results of the present section for application to one-dimensional structures and outline a method for obtaining exact solutions of the resulting integral equations.

4. APPLICATION TO CONTINUOUS BEAMS

The system of integral equations (3.6) takes on a specially simple form in the case of one-dimensional structures, for example continuous beams. The resulting integral equations have degenerate kernels and can be solved exactly.

Consider a uniform continuous beam with N spans and *elastic* bending stiffness B. Denote by l_i the length of the ith member and by $M(\zeta_i)$ and $\varkappa(\zeta_i)$ the bending moment and curvature at the section ζ_i of the ith member, respectively. Consider a loading at a given rate $p^{\cdot}(\zeta_i)$ from a given elasto–plastic state, and denote the corresponding rates of moments, elastic curvatures, and plastic curvatures by $M^{\cdot}(\zeta_i)$, $\varkappa^{\cdot}_e(\zeta_i)$, and $\varkappa^{\cdot}_p(\zeta_i)$, respectively. The equations of equilibrium are†

$$\frac{d^2 M^{\cdot}(\zeta_i)}{d\zeta_i^2} = -p^{\cdot}(\zeta_i); \qquad i = 1, 2, \ldots N, \tag{4.1a}$$

or

$$\frac{d^2}{d\zeta_i^2}\{B\varkappa^{\cdot}(\zeta_i)\} = -p^{\cdot}(\zeta_i) + \frac{d^2}{d\zeta_i^2}\{B\varkappa^{\cdot}_p(\zeta_i)\}, \tag{4.1b}$$

where $\varkappa^{\cdot}(\zeta_i) = \varkappa^{\cdot}_e(\zeta_i) + \kappa^{\cdot}_p(\zeta_i)$ is the curvature-rate, and $M^{\cdot}(\zeta_i) = B\varkappa^{\cdot}_e(\zeta_i)$ by Hooke's law.

Let $G(z_i; \zeta_j)$ denote the bending moment at section z_i of the ith member of the elastic structure due to a unit load applied at section ζ_j of its jth member. Using the method of Section 2 and equation (4.1b), we may write

$$B\varkappa^{\cdot}(z_i) = \sum_{j=1}^{N} \int_0^{l_j} p^{\cdot}(\zeta_j) G(z_i; \zeta_j)\, d\zeta_j - \sum_{j=1}^{N} \int_0^{l_j} [B\varkappa^{\cdot}_p(\zeta_j)]'' G(z_i; \zeta_j)\, d\zeta_j, \tag{4.2}$$

where primes denote differentiation with respect to the argument ζ_j. The first term in the right side of (4.2) would be the rate of moment at z_i due to the rate of loading $p^{\cdot}$ if the structure were to undergo no plastic deformations. We denote this by $M^{\cdot}_e(z_i)$. The second term on the right side of (4.2) can be integrated by parts, and noting that

$$-\lim_{\varepsilon \to 0}\left[B\varkappa^{\cdot}_p(\zeta_j)\frac{\partial G(z_i; \zeta_j)}{\partial \zeta_j}\right]_{\zeta_j = z_i - \varepsilon}^{\zeta_j = z_i + \varepsilon} = B\varkappa^{\cdot}_p(z_i),$$

equation (4.2) may be written as

$$M^{\cdot}(z_i) = M^{\cdot}_e(z_i) + \sum_{j=1}^{N} \int_0^{l_j} L(z_i; \zeta_j)\varkappa^{\cdot}_p(\zeta_j)\, d\zeta_j, \tag{4.3}$$

where

$$L(z_i; \zeta_j) = -B\frac{\partial^2 G(z_i; \zeta_j)}{\partial \zeta_j^2}$$

is Green's function defining the elastic bending moment at section z_i of the ith member of an equivalent elastic structure with the bending stiffness B caused by a unit relative rotation in an imagined hinge at section ζ_j of the jth member.

† No sum is implied on repeated indices in this section; the summation symbol Σ will be used to denote this operation.

For a structure that consists of an elastic, workhardening material, we have

$$\dot{\varkappa}_p = \begin{cases} A\dot{M} & \text{if} \quad |M| \geq Y \quad \text{and} \quad M\dot{M} > 0 \\ 0 & \text{if} \quad |M| < Y \quad \text{or} \quad |M| > Y \quad \text{and} \quad M\dot{M} < 0, \end{cases} \tag{4.4}$$

where the scalar A may depend on the considered plastic state, and where Y denotes the yield-point bending moment [8]. Substitution from (4.4) into (4.3) now yields

$$\dot{M}(z_i) = \dot{M}_e(z_i) + \sum_{j=1}^{N} \int_0^{l_j} L(z_i; \zeta_j) A(\zeta_j) \dot{M}(\zeta_j) \, d\zeta_j; \qquad i = 1, 2, \ldots, N, \tag{4.5}$$

which is the counterpart of system (3.6) for one-dimensional structures in pure bending.

In (4.5), Green's functions $L(z_i; \zeta_j)$ are linear functions in both variables z_i and ζ_j [9]. We may therefore write

$$L(z_i; \zeta_j) = (z_i a_{ij} + b_{ij}) + (z_i a'_{ij} + b'_{ij}) \zeta_j, \tag{4.6}$$

where a_{ij}, b_{ij}, a'_{ij}, and $b'_{ij} = a_{ij}; i, j = 1, 2, \ldots, N$, are constants. Equations (4.5) may thus be written as

$$\dot{M}(z_i) = \dot{M}_e(z_i) + \sum_{j=1}^{N} (z_i a_{ij} + b_{ij}) C_j + (z_i a'_{ij} + b'_{ij}) C'_j, \tag{4.7}$$

where

$$C_j = \int_0^{l_j} A(\zeta_j) \dot{M}(\zeta_j) \, d\zeta_j, \qquad C'_j = \int_0^{l_j} \zeta_j A(\zeta_j) \dot{M}(\zeta_j) \, d\zeta_j. \tag{4.8}$$

We now multiply both sides of (4.7) first by $A(z_i)$ and then by $z_i A(z_i)$ and integrate each of the resulting equations with respect to z_i over the plastic regions of the ith member to obtain

$$\sum_{j=1}^{N} (\delta_{ij} + h_{ij}) C_j + \sum_{j=1}^{N} h'_{ij} C'_j = f_i, \tag{4.9a}$$

$$\sum_{j=1}^{N} k_{ij} C_j + \sum_{j=1}^{N} (\delta_{ij} + k'_{ij}) C'_j = g_i, \tag{4.9b}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}, \qquad h_{ij} = -\int_0^{l_i} (a_{ij} z_i + b_{ij}) A(z_i) \, dz_i,$$

$$h'_{ij} = -\int_0^{l_i} (a'_{ij} z_i + b'_{ij}) A(z_i) \, dz_i, \qquad k_{ij} = -\int_0^{l_i} (a_{ij} z_i^2 + b_{ij} z_i) A(z_i) \, dz_i,$$

$$k'_{ij} = -\int_0^{l_i} (a'_{ij} z_i^2 + b'_{ij} z_i) A(z_i) \, dz_i$$

$$f_i = \int_0^{l_i} \dot{M}_e(z_i) A(z_i) \, dz_i, \quad \text{and} \quad g_i = \int_0^{l_i} z_i \dot{M}_e(z_i) A(z_i) \, dz_i. \tag{4.10}$$

The $2N$ linear equations (4.9) may now be solved for the $2N$ constants C_j and C'_j; $j = 1, 2, \ldots, N$, and the results may then be substituted into (4.7) yielding the exact solution of this system of integral equations. Note that in equations (4.10) the integrals are to be carried out on the plastic zones where the corresponding $A(z_i)$ is non-zero. If for a given rate of loading no unloading takes place, the values of $A(z_i)$; $i = 1, 2, \ldots, N$, are known and the integrations in (4.10) can be performed explicitly, resulting in a complete solution of the problem. On the other hand, if some elements that are in a plastic state undergo unloading, then the coordinates of the leading head of the corresponding new plastic zones can be taken as unknown parameters. Now, since at the new elastic–plastic interface we must have $\dot{M} = 0$, these unknown parameters can readily be calculated.

To illustrate some of the general results obtained in this section, let us consider a uniform beam that is built in at both ends and has a length l and an elastic bending stiffness B. If z and ζ measure distances from the left support, then the Green function is given by

$$L^*(z^*;\zeta^*) = (6z^*-4)-(12z^*-6)\zeta^*, \tag{4.11}$$

where the dimensionless quantities

$$L^*(z^*;\zeta^*) = \frac{l}{B}L\left(\frac{z}{l};\frac{\zeta}{l}\right), \qquad z^* = \frac{z}{l},$$

and $\zeta^* = \zeta/l$ are used.

Let $A(z) = \psi$ be a constant (a bi-linear moment–curvature relation), and consider an elasto–plastic state of the beam that corresponds to plastic regions in the intervals $0 \le z^* \le t^*$ and $1-t^* \le z^* \le 1$. Consider an additional loading that consists of a monotonically increasing, uniformally distributed lateral pressure which is applied at the rate of $\mathrm{d}p/\mathrm{d}p \equiv \dot{p} = 1$; thus, a superposed dot will denote differentiation with respect to p. Since for this additional loading, unloading may occur, we assume that the new plastic regions are in the intervals $0 \le z^* \le t$ and $1-t \le z^* \le 1$, where t is the root of equation $\dot{M}(t) = 0$ or equals t^* depending on whether $M(t^*)\dot{M}(t^*)$ is negative (unloading occurs) or positive (no unloading occurs).

With Green's function defined by (4.11), (4.5) reduces to

$$\begin{aligned} \dot{M}(z^*) &= \dot{M}_e(z^*)+\int_0^1 \psi[(6z^*-4)-(12z^*-6)\zeta^*]\dot{M}(\zeta^*)\,\mathrm{d}\zeta^* \\ &= \dot{M}_e(z^*)-2\psi\int_0^t \dot{M}(\zeta^*)\,\mathrm{d}\zeta^*, \end{aligned} \tag{4.12}$$

where $\dot{M}$ is the dimensionless moment-rate, and for the considered loading,

$$\dot{M}_e(z^*) = \tfrac{1}{2}(z^*-z^{*2}-\tfrac{1}{6}).$$

Integrating both sides of (4.12) with respect to z^* from zero to t we obtain

$$\dot{M}(z^*) = \tfrac{1}{2}(z^*-z^{*2}-\tfrac{1}{6})-\frac{\psi(3t^2-2t^3-t)}{6(1+2\psi t)}. \tag{4.13a}$$

where t is given by

$$t = \begin{cases} t^* & \text{if } M(t^*)\dot{M}(t^*) > 0, \\ \text{least root of } t^3+\dfrac{3(1-\psi)}{4\psi}t^2-\dfrac{3}{4\psi}t+\dfrac{1}{8\psi} = 0 & \text{if } M(t^*)\dot{M}(t^*) < 0. \end{cases} \tag{4.13b}$$

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Абстракт—Выводится система интегральных уравнений для скорости изменения поля напряжения, которая соответствует заданной скорости нагрузки из заданного упруго-пластического состояния в твердом теле с упрочнением. Определяется приближенный метод решения зтих интегральных уравнений для трех- и двух-мерных упруго-пластических задач. Результаты поименяются к одномерным конструкциям, таким как напр. непрерывные балки. Определяются решения в явном виде.